Notes:

Homework : Due next Tuesday.

Last class day: Weds. Dec 8.

Final Exam: Friday Dec. 10, 12:30 PM, will be comprehensive. Formula sheet allowed, similar to previous exam. More details later this week.

Recall, Fermi-Dirac statistics & **Fermi ideal gas**

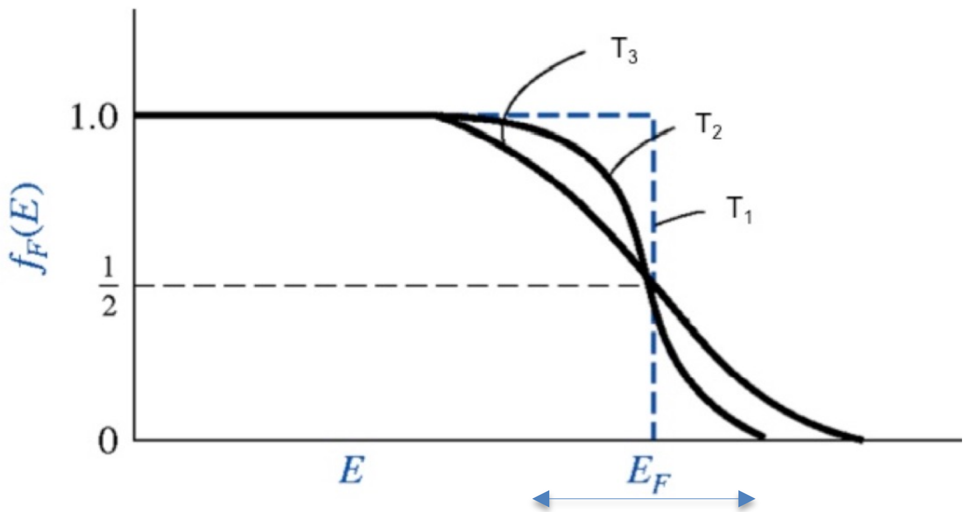
$$\varepsilon = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$$

Weak or non-existing interactions

$$n_{FD} = \frac{1}{(1 + e^{(\varepsilon_i - \mu)/kT})} = f(\varepsilon_i)$$

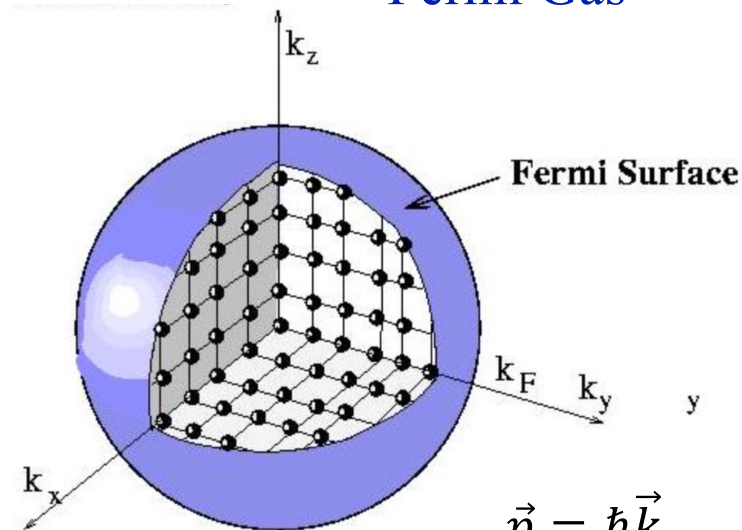
Fermi-Dirac distribution

"Fermi function"



10-90% width ≈ $4kT$

"Fermi Gas"



$\vec{p} = \hbar\vec{k}$

- Metals: electron gas typically *strongly degenerate*.
- White dwarf stars: degenerate *electron Fermi gas*, nuclei ≈ classical gas
- Neutron stars: neutrons (spin ½) strongly degenerate.

Fermi-Dirac statistics & Fermi ideal gas

$$n_{FD} = \frac{1}{(1 + e^{(\varepsilon_i - \mu)/kT})} = f(\varepsilon_i)$$

Fermi-Dirac distribution

"Fermi function"

$$\langle N \rangle = \sum_i f(\varepsilon_i)$$

Total # particles

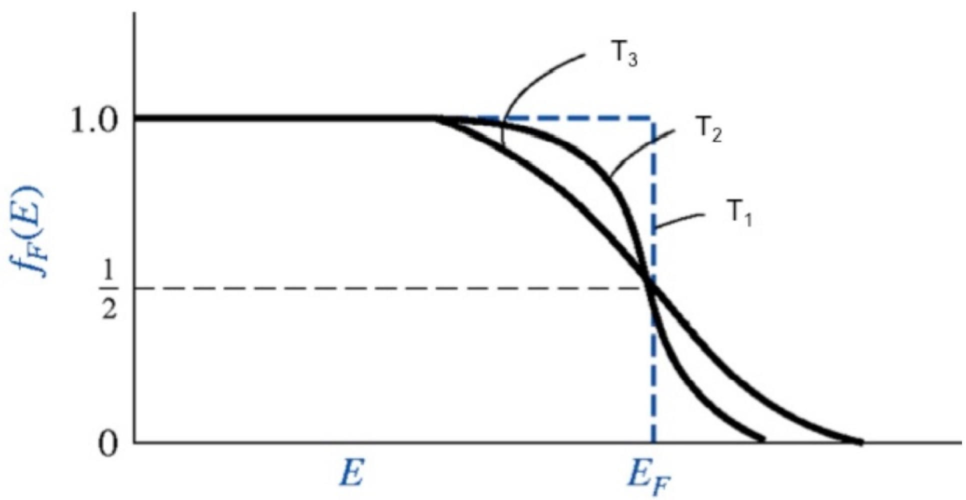
$$U = \sum_i \varepsilon_i f(\varepsilon_i)$$

Total (average) energy

For sums, use 3D density of states:

$$D(\varepsilon) = \frac{2V}{(4\pi^2)} \left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\varepsilon}$$

We saw this before; it is in chapter 18 except missing a factor of 2 for spin.



Summations (for spin-1/2)

$$\sum_i [X(\varepsilon_i)] \Longrightarrow \int D(\varepsilon)[X(\varepsilon)]d\varepsilon$$

Any function that depends on the energy

g

$$D(\varepsilon) = \frac{2V}{(4\pi^2)}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\varepsilon}$$

Density of states, ideal gas
(includes $2s+1$ spin term)

Specific cases:

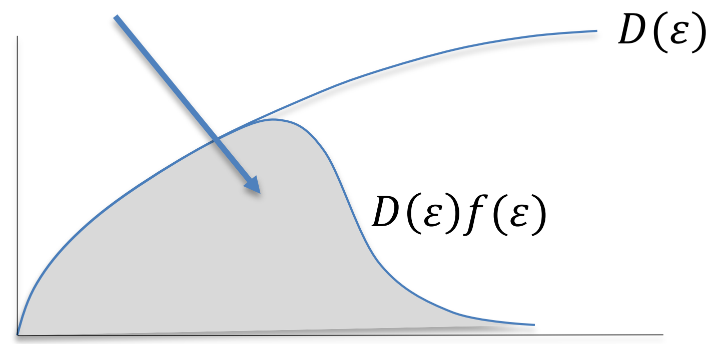
$$U = \sum_i \varepsilon_i f(\varepsilon_i)$$

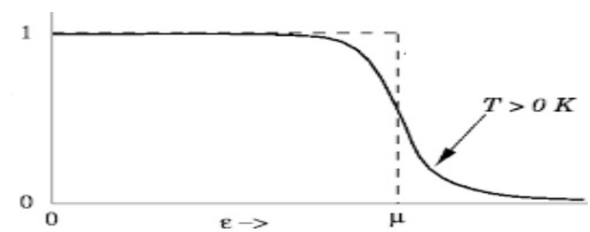
ideal gas,
large N

$$U = \int \varepsilon D(\varepsilon) f(\varepsilon) d\varepsilon$$

$N = \int D(\varepsilon) f(\varepsilon) d\varepsilon \qquad [f(\varepsilon) = n(\varepsilon)]$

is the area.





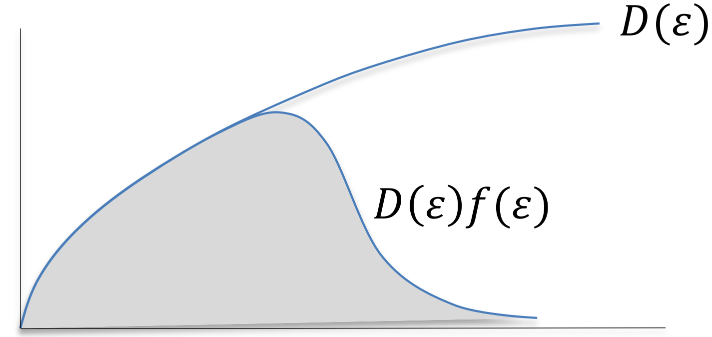
Fermi gas:

$$D(\varepsilon) = \frac{V}{(2\pi^2)}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\varepsilon}$$

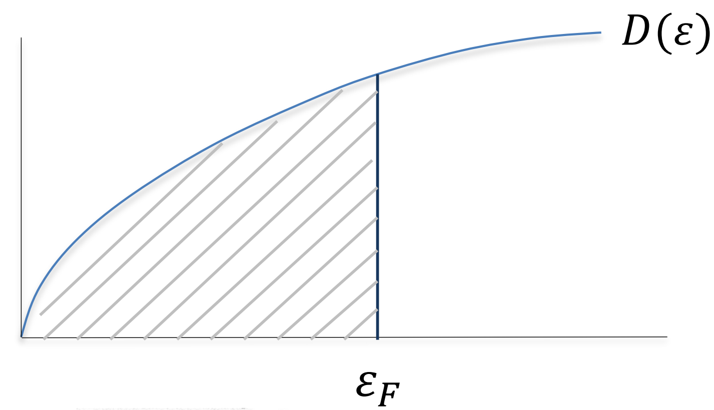
& note, $D(\varepsilon_F) = \frac{3N}{2\varepsilon_F}$

$$N = \int D(\varepsilon)f(\varepsilon)d\varepsilon$$

$$U = \int \varepsilon D(\varepsilon)f(\varepsilon)d\varepsilon$$

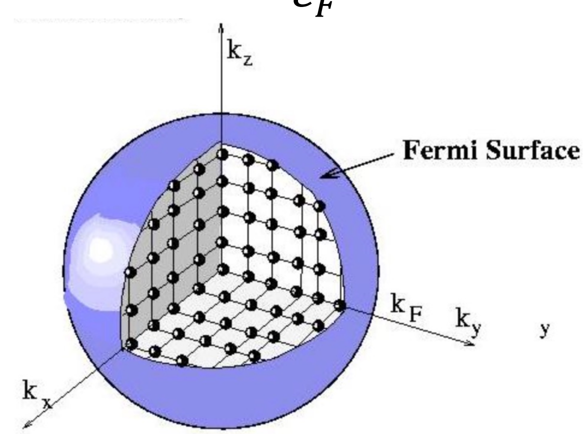


T = 0 results



$$\varepsilon_F = \frac{\hbar^2}{(2m)}\left(3\pi^2\frac{N}{V}\right)^{2/3}$$

Fermi Energy defined as $T = 0$ chemical potential



Fermi gas:

Metal, $T_F \sim$ 10,000's K

White dwarf, $T_F \sim 10^{10}$ K (T ~ 10^5 K)

Neutron star, $T_F \sim 10^{12}$ K (T ~ $10^6 - 10^{11}$ K)

Semiconductor, typically
nondegenerate

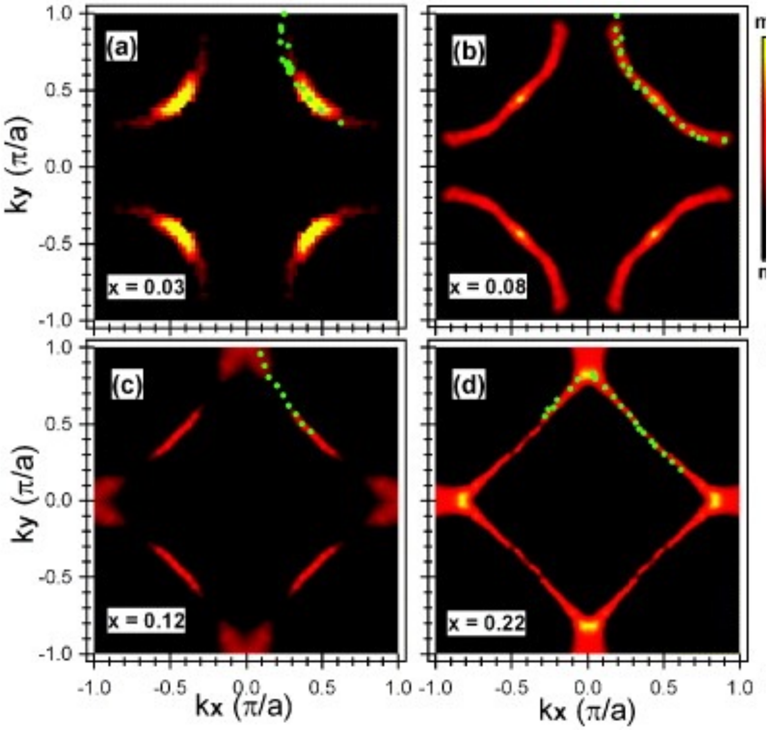
T = 0 results

$$\varepsilon_F = \frac{\hbar^2}{(2m)}\left(3\pi^2\frac{N}{V}\right)^{2/3}$$ Fermi Energy

$$k_F = \left(3\pi^2\frac{N}{V}\right)^{1/3}$$ Fermi wavevector

$$T_F = \frac{\hbar^2}{k_B(2m)}\left(3\pi^2\frac{N}{V}\right)^{2/3}$$ Fermi temperature

$$p_F = \hbar\left(3\pi^2\frac{N}{V}\right)^{1/3}$$ Fermi momentum

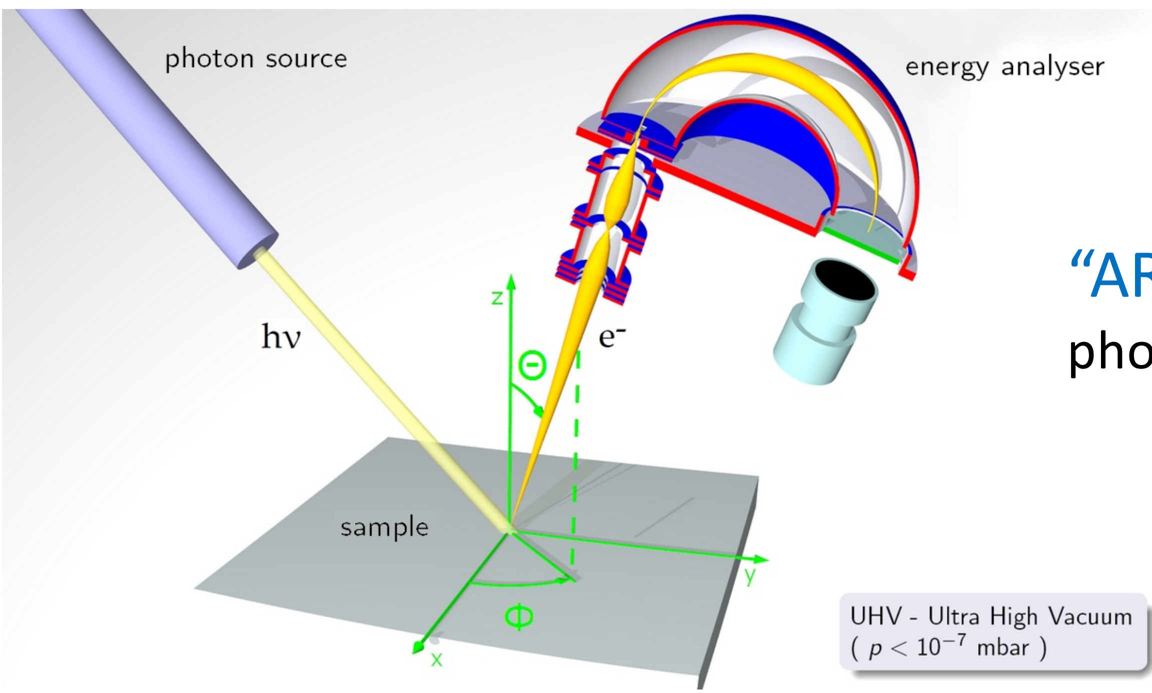


$La_{1-x}Sr_xCuO_4$ superconductor

Razzoli et al. New J. Physics 2010

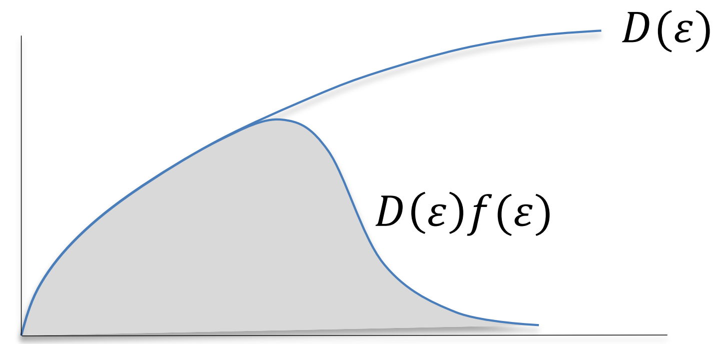
Photoemission: can detect momentum distribution of electrons

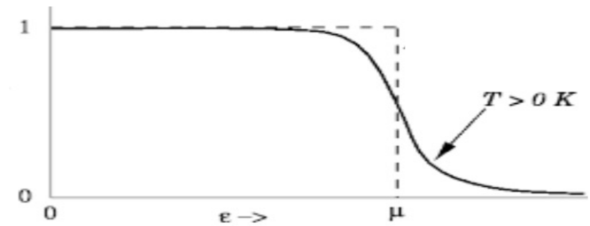
Detects Fermi sphere for simple metal (sodium).



"ARPES" Angle-resolved photoemission spectroscopy

Fermi gases:





$$n_{FD} = \frac{1}{\left(1 + e^{(\varepsilon_i - \mu)/kT}\right)} = f(\varepsilon_i)$$

$$U = \int \varepsilon D(\varepsilon) f(\varepsilon) d\varepsilon$$

$$D(\varepsilon) = \frac{V}{(2\pi^2)} \left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\varepsilon}$$ spin-1/2

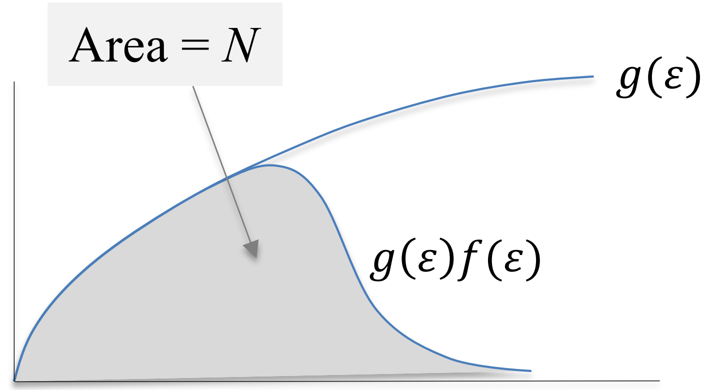
$T = 0$:

$$\varepsilon_F = \frac{\hbar^2}{(2m)} \left(3\pi^2 \frac{N}{V}\right)^{2/3}$$ Fermi Energy

$$\langle \varepsilon \rangle = \frac{3}{5} \varepsilon_F \qquad P = \frac{2}{5} \frac{N}{V} \varepsilon_F$$ degeneracy pressure

- good approximations as long as $T << T_F$
- P does <u>not</u> go to zero at $T = 0$. (Entropy *does*.)
- Metal conduction properties etc. controlled by thin slice $\sim kT$ at FS
- Degeneracy pressure supports white dwarf, neutron star vs. collapsing

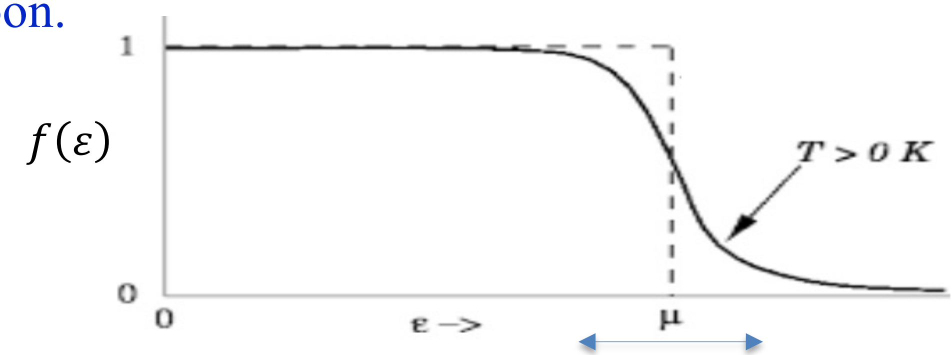
Fermi gas, $T \neq 0$



- Chemical potential: $\varepsilon_F \approx \mu, \quad T \ll T_F$

  ➢ $\mu \approx$ constant in degenerate regime. We will solve for T dependence soon.

  ➢ Energy $\propto T^2$ can see



90% width $\approx 4kT$

notation: ε_F is $T = 0$ limit, constant parameter determined by N/V.

- Crossover to MB regime, $\quad T \approx T_F \quad$:

$$\lambda_{th}^3 = \frac{8}{3\sqrt{\pi}}\left(\frac{V}{N}\right)$$

➢ Overlapping of thermal de Broglie wavelengths is (approximate) boundary of degenerate regime

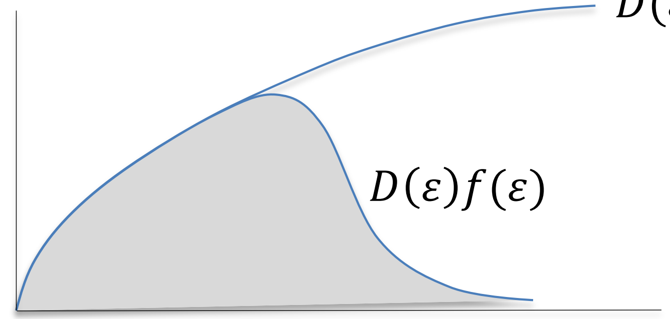
$$T_F = \frac{\hbar^2}{k_B(2m)}\left(3\pi^2\frac{N}{V}\right)^{2/3}$$

$$p_F = \hbar\left(3\pi^2\frac{N}{V}\right)^{1/3}$$

$$\left(\frac{h^2}{2\pi mkT}\right)^{1/2} = \lambda_{th}$$

Electron: ~30 Å at RT

T dependence:



$$n_{FD} = \frac{1}{(1 + e^{(\varepsilon_i - \mu)/kT})} = f(\varepsilon_i)$$

$$U = \int \varepsilon D(\varepsilon) f(\varepsilon) d\varepsilon$$

$T \neq 0$ cases **Sommerfeld expansion**

$\mu - \varepsilon_F \sim (\frac{T^2}{T_F^2})$:

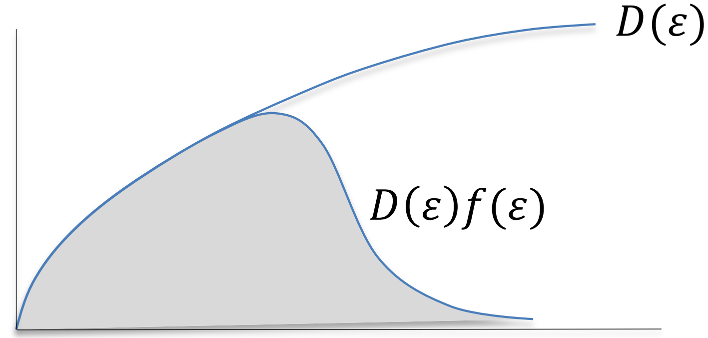
$$N = \int D(\varepsilon) f(\varepsilon) d\varepsilon = -\int \left[\int_{-\infty}^{\varepsilon} D(\varepsilon') d\varepsilon'\right] \frac{df}{d\varepsilon} d\varepsilon$$

by parts

$-\frac{df}{d\mu}$

T dependence:

$$n_{FD} = \frac{1}{(1 + e^{(\varepsilon_i - \mu)/kT})} = f(\varepsilon_i)$$



$$U = \int \varepsilon D(\varepsilon) f(\varepsilon) d\varepsilon$$

$T \neq 0$ cases **Sommerfeld expansion**

$\mu - \varepsilon_F \sim (\frac{T^2}{T_F^2})$:

$$N = \int D(\varepsilon) f(\varepsilon) d\varepsilon = -\int \left[\int_{-\infty}^{\varepsilon} D(\varepsilon') d\varepsilon'\right] \frac{df}{d\varepsilon} d\varepsilon$$

$\frac{df}{d\varepsilon} \to -\frac{df}{d\mu}$

by parts

$$N = \int \left[\int_{-\infty}^{\mu} D(\varepsilon') d\varepsilon'\right] \frac{df}{d\mu} d\varepsilon + \int [D(\mu)](\varepsilon - \mu) \frac{df}{d\mu} d\varepsilon + \int \left[\frac{dD(\mu)}{d\mu}\right] \frac{(\varepsilon - \mu)^2}{2} \frac{df}{d\mu} d\varepsilon + \cdots$$

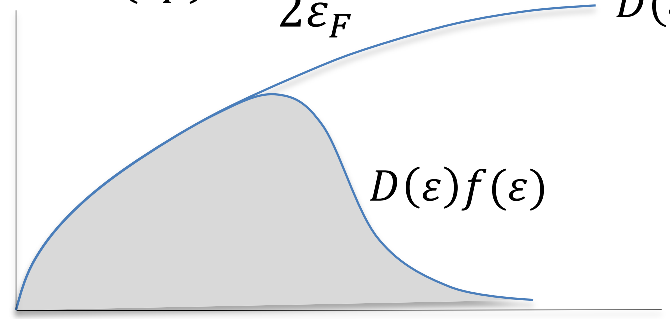
Taylor expand square bracket

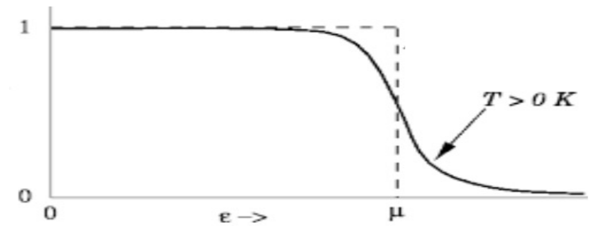
0, odd function

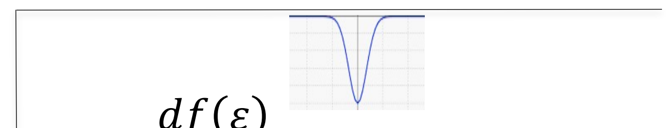
T dependence:

$$n_{FD} = \frac{1}{\left(1 + e^{(\varepsilon_i - \mu)/kT}\right)} = f(\varepsilon_i)$$

$$E = \int \varepsilon D(\varepsilon) f(\varepsilon) d\varepsilon$$







$$-\frac{df(\varepsilon)}{d\varepsilon} \approx \text{Dirac delta fn.}$$

$$N = \int D(\varepsilon) f(\varepsilon) d\varepsilon = -\int \left[\int_{-\infty}^{\varepsilon} D(\varepsilon') d\varepsilon'\right] \frac{df}{d\varepsilon} d\varepsilon \qquad -\frac{df}{d\mu}$$

by parts

$$N = \int \left[\int_{-\infty}^{\mu} D(\varepsilon') d\varepsilon'\right] \frac{df}{d\mu} d\varepsilon + \int [D(\mu)](\varepsilon - \mu) \frac{df}{d\mu} d\varepsilon + \int \left[\frac{dD(\mu)}{d\mu}\right] \frac{(\varepsilon - \mu)^2}{2} \frac{df}{d\mu} d\varepsilon + \cdots$$

Taylor expand square bracket

0, odd function

$$N \approx \int_{-\infty}^{\mu} D(\varepsilon') d\varepsilon' + (kT)^2 \frac{\pi^2}{6} \left[\frac{dD(\mu)}{d\mu}\right]$$

$$\left.\frac{\partial \mu}{\partial T}\right]_N = -\left.\frac{\partial N}{\partial T}\right]_\mu \Big/ \left.\frac{\partial N}{\partial \mu}\right]_T = -\frac{Vk^2 T 2 \frac{\pi^2}{6}\left[\frac{dD(\mu)}{d\mu}\right]}{VD(\mu) + (small)}$$

Fermi gases:

$$\mu \approx \varepsilon_F\left(1-\frac{\pi^2}{12}\left[\frac{T}{T_F}\right]^2\right)$$

similar expansion for energy:

$$-\frac{df}{d\mu}$$

$$E=\int \varepsilon D(\varepsilon)f(\varepsilon)d\varepsilon=-\int\left[\int_{-\infty}^{\varepsilon}\varepsilon' D(\varepsilon')d\varepsilon'\right]\frac{df}{d\varepsilon}d\varepsilon$$

by parts

$$E=\int\left[\int_{-\infty}^{\mu}\varepsilon' D(\varepsilon')d\varepsilon'\right]\frac{df}{d\mu}d\varepsilon+\int[\mu D(\mu)](\varepsilon-\mu)\frac{df}{d\mu}d\varepsilon$$

$$+\int\left[\frac{d[\mu D(\mu)]}{d\mu}\right]\frac{(\varepsilon-\mu)^2}{2}\frac{df}{d\mu}d\varepsilon+\cdots$$

Taylor expand square bracket

$$E\approx\int_{-\infty}^{\varepsilon_F}\varepsilon D(\varepsilon)d\varepsilon+[\varepsilon_F D(\varepsilon_F)](\mu-\varepsilon_F)+(kT)^2\frac{\pi^2}{6}D(\varepsilon_F)+(kT)^2\frac{\pi^2}{6}D(\varepsilon_F)\left.\frac{dD}{d\mu}\right]_{\varepsilon_F}$$

2 terms cancel

$$E\approx E_o+(kT)^2\frac{\pi^2 N}{4\varepsilon_F}$$