A three-dimensional lattice has an optical branch modeled by, \( \omega = \alpha \sqrt{k^2 + k_o^2} \), where \( \alpha \) is a constant and \( k_o \) is a cutoff wavevector; make the approximation of spherical symmetry, so that we assume the modes only occupy a region of \( k \)-space with \( 0 \leq k \leq k_o \), in all directions.

(a) Derive the density of modes, \( D(\omega) \).

(b) For one of the phonon modes with the highest frequency, find the average number of phonons in thermal equilibrium, in the limit as \( T \) goes to zero (leading non-zero term), and also in the limit of high temperature (again, expanding and identifying the leading term).
(2) A metal has an ellipsoidal energy band, given by,

\[ \epsilon(k) = \frac{\hbar^2}{2} \left( \frac{k_x^2}{m_x} + \frac{k_y^2}{m_y} + \frac{k_z^2}{m_z} \right), \]

where the three masses are different constant values. Assume this to be a simple orthorhombic lattice, of lattice constants \( a, b, \) and \( c \) (in \( x, y, z \) directions). Also for purposes of this problem neglect any band-bending or energy gaps at the zone boundaries; assume the expression above holds for the entire Brillouin zone.

(a) Derive the semiclassical velocity (or, its components), as a general function of \( \tilde{k} \).

(b) For an electric field of magnitude \( E_0 \) in the \( x \) direction, find an expression for \( \tilde{k} \) as a function of time, starting at a general position \( \tilde{k}_o \), and explain what happens at zone boundaries. Assume no scattering.

(c) Write an expression for \( \tilde{v} \) as a function of time, assuming that \( \tilde{k}_o = (0, \frac{\pi}{2b}, 0) \).

(d) Sketch the real-space path for this electron. This path contains a periodic feature; find the real-space dimension, in the \( y \)-direction.
(3) A simple-cubic crystal, lattice constant $a$, has an electron energy band,

$$\epsilon = C(\cos k_x a - \cos k_y a + 2 \cos k_z a),$$

where $C$ is a positive constant.

(a) Find a position in $k$-space which will be the center of a hole pocket for this band.

(b) Find the effective mass tensor at the position of this hole pocket.
(4) Consider a semiconductor with a single electron pocket at the bottom of the conduction band, having effective mass $m^*$ (isotropic). Assume that for all relevant energies, the effective mass approximation works so that the band-edge energy is quadratic in $k$, for all $k$ (so that the band can go up to infinity at its upper limit).

(a) Work out the velocity as a function of the energy, $\epsilon - \epsilon_c$, near this band edge, using the effective mass approximation.

(b) Now derive an expression for the average square velocity, $\langle v^2 \rangle$, per volume, for electrons in thermal equilibrium in this band. Assume that the chemical potential is far enough below the band edge so that the Fermi function becomes a simple exponential. You will have an integral with infinite limits; factor out the temperature so that the integral becomes a unitless integral.

(c) The chemical potential can be eliminated from your expression above by obtaining an expression for the electron density, $n$. Do this, and as a result show that $\langle v^2 \rangle$ per electron has a simple expression independent of the chemical potential.